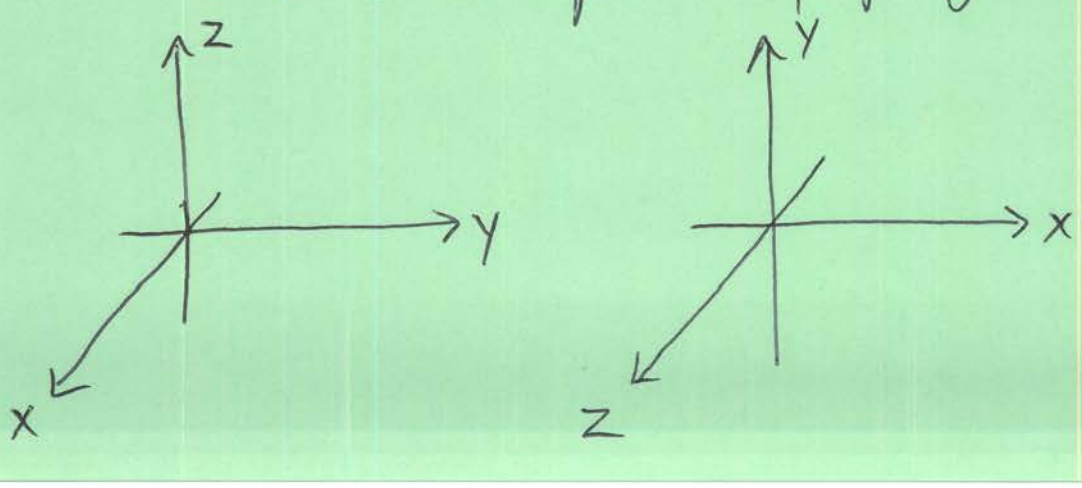
Lecture 1

12.1 - 3-dimensional Coordinate System

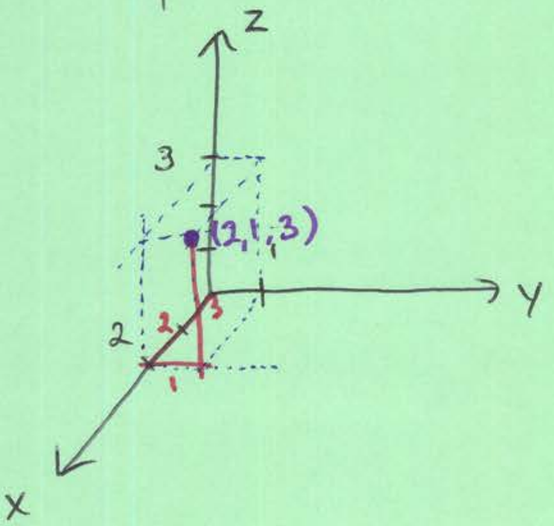
The 3-dimensional coordinate system we use are coordinates on $\mathbb{R}^3$. The coordinate is presented as a triple of numbers: (a, b, c). In the Cartesian coordinate system we have an origin, $(0,0,0)$, and three axes: the x-, y-, and z- axes. These 3 axes are perpendicular to each other and their positive directions satisfy the "right hand rule": point your index finger on your right hand along the x-axis, curl it toward the y-axis, then your "thumb up" will point along the z-axis. Examples of properly drawn axes are:



(arrows denote positive direction)

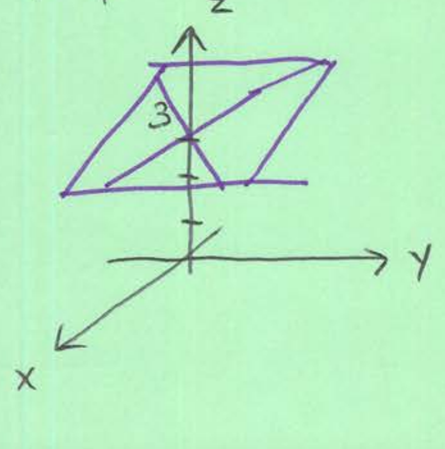
To locate the point P which has coordinates (a,b,c): move a units in the x-direction, b in the y-direction, and c in the z-direction.

Ex: (a) The point $(2,1,3)$ can be graphed as



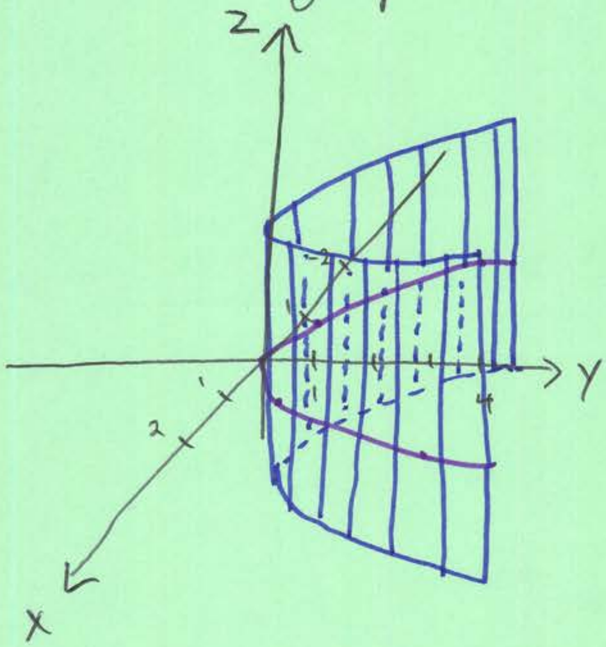
(b) What would the equation $z=3$ represent in $\mathbb{R}^3$?

The only restriction here is that $z=3$, so any point of the form $(x,y,3)$ satisfies this. This is a plane, parallel to the xy-plane, at "height" $z=3$:



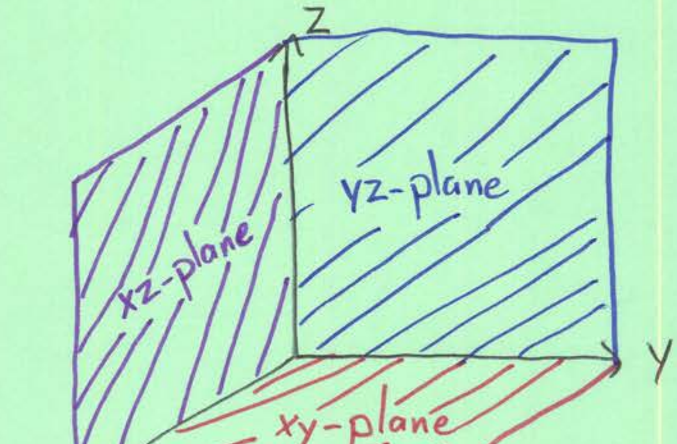
© How about $y = x^2$?

In the xy-plane, this is just a parabola... but in $\mathbb{R}^3$, this equation gives us no restriction on z, so the graph of the equation is



The <u>coordinate planes</u> are the xy-, xz-, and yz-planes, which are represented by $z=0$, $y=0$, and $x=0$ respectively.

<u>Graphically:</u>



We can also talk about "projecting" onto the coordinate planes. This is done by setting the appropriate coordinate to 0.

The <u>projection</u> of (a,b,c) onto the:

- xy-plane is $(a,b,0)$
- xz-plane is $(a,0,c)$
- yz-plane is $(0,b,c)$

Just as in the plane, we can talk about the distance between points in space. Applying the Pythagorean Theorem twice, we arrive at the distance formula.

<u>Distance Formula in $\mathbb{R}^3$</u>: Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$. The distance from P_1 to P_2 is

$$\boxed{d(P_1,P_2)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

(the book uses $|P_1P_2|$ instead of $d(P_1,P_2)$.)

Consider the point (h,k,ℓ). Suppose we want an equation for the collection of points which are a distance r away from (h,k,ℓ). Using the distance formula, we know any point (x,y,z) satisfying this criteria satisfies:

$$r = d\big((h,k,\ell),(x,y,z)\big) = \sqrt{(x-h)^2+(y-k)^2+(z-\ell)^2}$$

This set of points is the <u>sphere</u> with <u>radius</u> r and <u>center</u> (h,k,ℓ). Squaring both sides of the equation, we arrive at a more friendly equation for the <u>sphere</u>:

$$\boxed{(x-h)^2+(y-k)^2+(z-\ell)^2=r^2}$$

<u>Ex</u>: The region described by the inequalities

$x^2+y^2+z^2 \leq 4$ & $x^2+y^2 \geq 1$

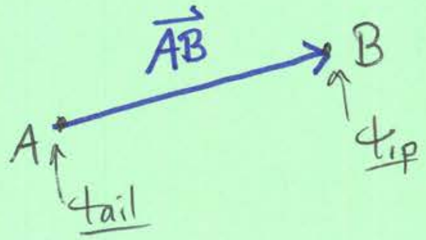
looks like a solid ball of radius 2, centered at the origin with a hole of radius 1 drilled through it along the z-axis.

12.2 - Vectors

Def: A vector is an object with direction and magnitude.

There is one exception to this definition, the zero vector, $\vec{0}$, which has magnitude 0 and no specified direction.

Suppose a particle moves from a point A to a point B along a straight line. Then the displacement vector, written $\vec{AB}$, can be visualized as an arrow from A to B, visually:

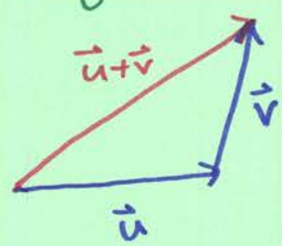
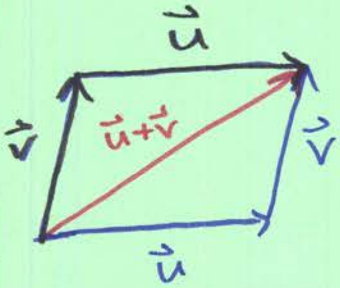
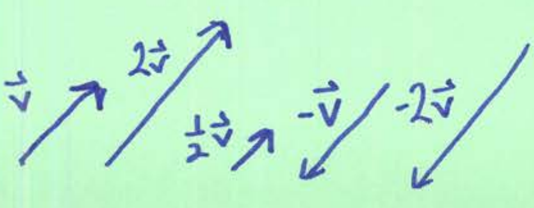


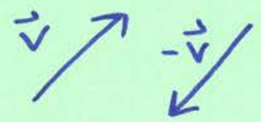
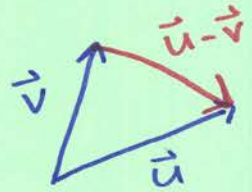
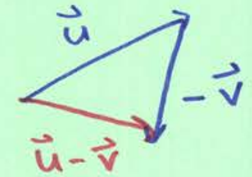
If the points have coordinates $A=(a_1, a_2, a_3)$ & $B=(b_1, b_2, b_3)$, we can represent $\vec{AB}$ as

$$\vec{AB} = B - A = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

(this works for points in $\mathbb{R}^2$ as well)

Vector operations : (Everything here is written for vectors in $\mathbb{R}^2$, but works in $\mathbb{R}^3$ as well)

Operation	Visually	Algebraically
Vector Addition $\vec{u}+\vec{v}$	Place the tail of $\vec{v}$ on the tip of $\vec{u}$, then $\vec{u}+\vec{v}$ starts at the tail of $\vec{u}$, and ends at the tip of $\vec{v}$: Triangle Law :  Parallelogram Law:  	$\vec{u}=\langle u_1, u_2\rangle$ $\vec{v}=\langle v_1, v_2\rangle$ $\vec{u}+\vec{v}=\langle u_1+v_1, u_2+v_2\rangle$
Scalar Multiplication $c\vec{v}$	Scale the size of $\vec{v}$ by $\lvert c\rvert$ and point $\vec{v}$ in the opposite direction if $c<0$  	$c\in\mathbb{R}$, $\vec{v}=\langle v_1, v_2\rangle$ $c\vec{v}=\langle cv_1, cv_2\rangle$

<u>Negative</u> $-\vec{v}$	$-\vec{v}$ points in the opposite direction 	$\vec{v} = \langle v_1, v_2 \rangle$ $-\vec{v} = \langle -v_1, -v_2 \rangle$
Vector Subtraction $\vec{u} - \vec{v}$	 	$\vec{u} = \langle u_1, u_2 \rangle$ $\vec{v} = \langle v_1, v_2 \rangle$ $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ $= \langle u_1 - v_1, u_2 - v_2 \rangle$

Magnitude of a Vector

In $\mathbb{R}^3$, $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Algebraic Properties of Vectors:

1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3) $\vec{a} + \vec{0} = \vec{a}$
4) $\vec{a} + (-\vec{a}) = \vec{0}$
5) $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6) $(c+d)\vec{a} = c\vec{a} + d\vec{a}$
7) $(cd)\vec{a} = c(d\vec{a})$
8) $1\vec{a} = \vec{a}$

Given any vector $\vec{v} = \langle a, b, c \rangle$, using the rules above, we can write

$$\vec{v} = \langle a, b, c \rangle = a\langle 1,0,0 \rangle + b\langle 0,1,0 \rangle + c\langle 0,0,1 \rangle$$
$$= a\hat{i} + b\hat{j} + c\hat{k}$$

$\hat{i} = \langle 1,0,0 \rangle$, $\hat{j} = \langle 0,1,0 \rangle$, and $\hat{k} = \langle 0,0,1 \rangle$ are called the <u>standard basis vectors</u> in $\mathbb{R}^3$ (likewise, $\hat{i} = \langle 1,0 \rangle$ & $\hat{j} = \langle 0,1 \rangle$ are the standard basis vectors for $\mathbb{R}^2$). The coefficients of $\hat{i}, \hat{j},$ & $\hat{k}$ are called the <u>components</u> of $\vec{v}$.

<u>Def</u>: A <u>unit vector</u> is a vector of magnitude 1.

(I will usually denote unit vectors with a ^ instead of →.)

Given a vector $\vec{v} \neq \vec{0}$, one can find the <u>unit vector in the direction of $\vec{v}$</u> by multiplying by $\frac{1}{\|\vec{v}\|}$, i.e.,

$$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$$

is a unit vector in the direction of $\vec{v}$.

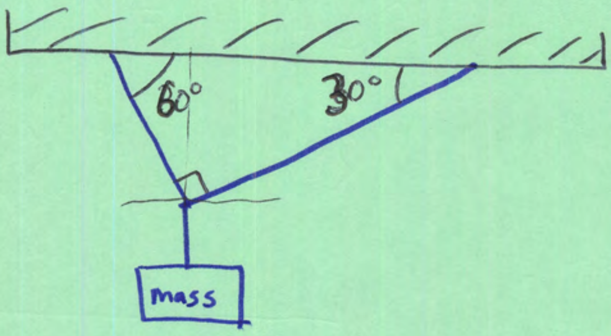
Given a vector's magnitude and direction (angle it makes with positive x-axis) we can recover the vector: If $\vec{v}$ is the vector, $\|\vec{v}\|$ its magnitude, and direction θ, $\vec{v}$ can be written:

$$\vec{v} = \|\vec{v}\|\cos\theta\,\hat{i} + \|\vec{v}\|\sin\theta\,\hat{j}$$

Of course, this is only true for 2 dimensional vectors. The procedure is a bit different in higher dimensions.

An Application

Ex: Suppose we have a 100 kg suspended from the ceiling as depicted:



Using $g = 9.8\ m/s^2$ for acceleration due to gravity, find the tension in each cable.

Sol: Let $\vec{T}_1$ & $\vec{T}_2$ denote the tensions in the left & right cables, resp. Let $\vec{w}$ denote the weight vector. Then $\vec{w} = \langle 0, -mg \rangle = \langle 0, -980 \rangle$ By Newton's 3rd law, the sum of $\vec{T}_1$, $\vec{T}_2$, and $\vec{w}$ must be $\vec{0}$ since the weight is not in motion, i.e., $\vec{T}_1 + \vec{T}_2 + \vec{w} = \vec{0}$. In components, we have 2 equations:

$$\begin{cases} \|\vec{T}_1\| \cos 120° + \|\vec{T}_2\| \cos 30° + 0 = 0 \\ \|\vec{T}_1\| \sin 120° + \|\vec{T}_2\| \sin 30° - 980 = 0 \end{cases}$$

$$\begin{cases} -\frac{1}{2}\|\vec{T}_1\| + \frac{\sqrt{3}}{2}\|\vec{T}_2\| = 0 \\ \frac{\sqrt{3}}{2}\|\vec{T}_1\| + \frac{1}{2}\|\vec{T}_2\| = 980 \end{cases} \Rightarrow \begin{cases} \sqrt{3}\,\|\vec{T}_2\| = \|\vec{T}_1\| & (1) \\ \sqrt{3}\,\|\vec{T}_1\| + \|\vec{T}_2\| = 1960 & (2) \end{cases}$$

Plug (1) into (2):

$$3\|\vec{T}_2\| + \|\vec{T}_2\| = 4\|\vec{T}_2\| = 1960 \Rightarrow \|\vec{T}_2\| = 490$$

$$\overset{(1)}{\Rightarrow} \|\vec{T}_1\| = \sqrt{3}\cdot 490 = 490\sqrt{3}$$

◊

12.3 - Dot Product

We've discussed how to add, subtract, and multiply vectors by a scalar, but what about multiplying vectors? Should it produce a number, or a vector? This first product will produce a scalar:

<u>Dot Product</u>: For $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the <u>dot product</u> of $\vec{u}$ and $\vec{v}$ is

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

The dot product is sometimes called the <u>scalar</u> or <u>inner product</u>. (The dot product for 2D vectors is defined similarly.)

Properties of the Dot Product

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors and c a scalar.

1) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$ 2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ 3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

4) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot b) = \vec{a} \cdot (c\vec{b})$ 5) $\vec{0} \cdot \vec{a} = 0$

Lecture 2

Suppose the angle between two vectors $\vec{u}$ & $\vec{v}$ is θ, then another interpretation of the dot product is:

$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos \theta}$$

This can be reversed to find the angle between two vectors $\vec{u}$ & $\vec{v}$:

$$\boxed{\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right)}$$

Two vectors are called <u>perpendicular</u> or <u>orthogonal</u> if their dot product is 0 (i.e., $\theta = 90°$)

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$$